Research Statements

Fei Si

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I am broadly interested in algebraic geometry and the related topics. Currently, I am working on the following topic:

0.1 Birational geometry of moduli space of K3 surfaces

0.1.1 Current work

Let $\mathcal{F}_g$ be the coarse moduli space of the quasi-polarised K3 surfaces (S, L) of degree $L^2 = 2g - 2 > 0$. From Hodge theoretic construction, the global Torelli theorem says the moduli space $\mathcal{F}_g$ is isomorphic to a Shimura variety of orthogonal type. From GIT side, $\mathcal{F}_g$ can be realised as a quotient space of a certain parameter space in the sense of Mumford's GIT. In low degree, $\mathcal{F}_g$ can also be viewed as moduli space of certain K-stable Del Pezzo surface pairs. It turns out these three view points will provide different compactifications. In joint works [4] with Francois Greer, Radu Laza, Zhiyuan Li, Zhiyu Tian, we extend the so-called Hassett-Keel-Loojigenga program to the moduli space of K3 surfaces with Mukai model, which is a log minimal model program (LMMP) on moduli space. By Mukai [7], for $g \leq 7$, a general K3 surface of genus g can be realised as a complete intersections of sections of a homogenous vector bundle on certain homogenous space, eg, a general K3 surface of genus $g = 4$ is zero locus of a section of vector bundle $E := \mathcal{O}_{\mathbb{P}^4}(2) \oplus \mathcal{O}_{\mathbb{P}^4}(3) \to \mathbb{P}^4$. These models provide a natural GIT compactification, say $\overline{\mathcal{F}}_g^{GIT}$.

Let $P_{d,h}$ be the primitive Noether-Lefschetz divisor on $\mathcal{F}_g$ parametrizing quasi-polarise K3 surfaces (S, L) of genus g with additional curve class $\beta \in H^2(S, \mathbb{Z})$ such that

$$\begin{array}{c|c|c} & L & \beta \\ \hline L & 2g-2 & d \\ \hline \beta & d & 2h-2 \end{array}.$$

We propose to study

$$\overline{\mathcal{F}}_g(s) = \text{Proj } R(\mathcal{F}_g^\circ, (\lambda_g + s\Delta)|_{\mathcal{F}_g^\circ})$$

Here $\mathcal{F}_g^\circ$ is the complement of certain Noether-Lefschetz divisors and Δ is a certain combination of Noether-Lefschetz divisors, see [4] for the complete list of our choice $\mathcal{F}_g^\circ$ and Δ for $g \leq 7$. We conjecture varying s there are wall-crossings which give a explicit resolution of birational maps of the two compactifiactions

$$\mathcal{F}_g^* \cong \overline{\mathcal{F}}_g(0) \dashrightarrow \overline{\mathcal{F}}_g(1) \cong \overline{\mathcal{F}}_g^{GIT}.$$

and modularity principles for the wall-crossing hold, that is, at each wall, the birational is a flip or divisorial contraction centered at shimura subvarieties (or its proper transformation) of $\mathcal{F}_g^*$. For $g = 3$, our model recover the model studied by Laza-O'Grady [5]. For $g = 4$, we give a complete conjecture

Conjecture 0.1 ([4]) *Let $\mathcal{F}_g^\circ$ be the complement of $P_{1,1}$ and $P_{2,1}$ and let $\Delta = P_{3,1}$. Then*

1. *the ring of sections*
$$R(\mathcal{F}_g^\circ, (\lambda_g + s\Delta)|_{\mathcal{F}_g^\circ}) \cong R(\mathcal{F}_g, \lambda_g + P_{1,1} + (4s+1)P_{2,1} + sP_{3,1}) \tag{1}$$
is finitely generated for $s \in [0,1] \cap \mathbb{Q}$.

2. *The interpolating models between $\overline{\mathcal{F}}_6(0)$ and $\overline{\mathcal{F}}_g(1)$ undergoes elementary birational transformations (flips or divisorial contractions) at the following walls*
$$s \in \{\frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \frac{1}{13}, \frac{1}{15}, \frac{1}{17}, \frac{1}{19}, \frac{1}{21}, \frac{1}{33}\}.$$
Let s_n be the slope of the n-th wall and set
$$\overline{\mathcal{F}}_g(s_n, s_{n+1}) = \overline{\mathcal{F}}_g(s), s \in (s_n, s_{n+1}).$$
and $E_i^\pm$ the exceptional divisors of the birational maps
$$\begin{tikzcd} \overline{\mathcal{F}}_g(s_{n-1}, s_n) \arrow[dr, "f_n^-"] & & \overline{\mathcal{F}}_g(s_n, s_{n+1}) \arrow[dl, "f_n^+"'] \\ & \overline{\mathcal{F}}_g(s_n) & \end{tikzcd} \tag{2}$$

3. *The exceptional divisor E_n^+ is the proper transformation of a Shimura subvariety $\mathcal{S}_n \subseteq \mathcal{F}_g$.*

Theorem 0.2 (Greer-Laza-Li-Si-Tian [4]) *Conjecture 0.1 holds when $0 \leq s \leq \frac{2}{3}$ or $s = 1$.*

Our results provide more evidence of modularity principle.

The birational geometry of a variety is controlled by its various cone, eg, its ample cone, effective cone. There are many re for M_g, see for a survey. For $\mathcal{F}_g$, few is known. Let $\mathrm{Eff}(\mathcal{F}_g^*)$ ($\mathrm{Eff}^{\mathrm{NL}}(\mathcal{F}_g^*)$) be the cone in $\mathrm{Pic}(\mathcal{F}_g^*)$ generated by effective divisor (rep. primitive Noether-Lefschetz divisors) a question asked by Peterson in [9] is

Question 0.3 1. *Is $\mathrm{Eff}(\mathcal{F}_g^*)$ finitely generated ?*

2. $\mathrm{Eff}(\mathcal{F}_g^*) = \mathrm{Eff}^{\mathrm{NL}}(\mathcal{F}_g^*)$ *?*

As a byproduct in our study of Hassett-Keel-Loojigenga program, we give a negative answer for question 2 of 0.3:

Theorem 0.4 (Greer-Laza-Li-Si-Tian [4]) *For* $g = 4$, $\mathrm{Eff}^{\mathrm{NL}}(\mathcal{F}_g^*) \subsetneq \mathrm{Eff}(\mathcal{F}_g^*)$.

Moreover, we also give plenty of examples extremal effective divisors:

Theorem 0.5 (Greer-Laza-Li-Si-Tian [4]) *The primitive Noether-Lefschetz divisors* $P_{d,h}$ *on* $\mathcal{F}_g^*$ *are extrmal provided* $|d^2 - 4(g-1)(h-1)| \leq \frac{15}{8}(g-1)$.

0.1.2 Future work

Investigate the the wall-crossing in 0.1 with the relation of wall-crossing in K-stability.

0.2 Explicit K-moduli space of log Fano surface pair

0.2.1 Current work

The algebraic construction of compact K-moduli space has attracted lots of attentions, eg, see [12] for the most recent survey. Especially, Ascher-DeVleming-Liu [1] establish a framework for the K-moduli space of log Fano pairs and wall-crossing phenomenon. But the explicit examples are rare. A project that I am working on is the following: Let $P_{d,\epsilon}$ be the moduli space of 2-dimensional K-semistable log Fano pair smoothing by $(X, \epsilon D)$ where X is a Del Pezzo surface of degree $d = (-K_X)^2$ and $D \in |-2K_X|$, $\epsilon \in (0, \frac{1}{2}) \cap \mathbb{Q}$.

Observe that a double covering $\phi : S \to X$ of Del Pezzo surface of degree d branched along a curve $D \in |-2K_X|$ will give a K3 surface $(S, \phi^*(-K_X); \tau)$ of genus $d-1$ with anti-involution τ. Using period map of K3 with anti-involution, we get Torelli theorem for the open subset $P^\circ_{d,\epsilon}$ of $P_{d,\epsilon}$ parametrizing smooth pairs. So we can identify $P^\circ_{d,\epsilon}$ as a open subset a shimura varieties. In work in progress, I will

1. give a explicit description of the wall-crossing for $d = 4, 5$ using the methods of VGIT.
2. Compute the volume of CM-line bundle on $P_{d,\epsilon}$ for $d = 5$ (note that degree 4 case has been computed by recently Tambasco), which corresponds to Weil-Peterson volume in differential geometry.
3. give a conjectural picture for the wall at $\epsilon = \frac{1}{2}$, which identify moduli spaces of log CY surface and certain arithemetic compactification of $P^\circ_{d,\epsilon}$ and try to prove it.

0.2.2 Future work

Let $\mathcal{P}_g$ moduli space of pairs (X, S) where X is a smooth Fano 3-fold with $(-K_X)^3 = 2g - 2$ and $S \in |-K_X|$ is a smooth divisor which produces a K3 surface $(S,(-K_X)|_S)$of genus of genus g. Denote W_g the moduli space of smooth Fano 3-folds of volume $(-K_X)^3 = 2g-2$. The forgetting

$$\begin{tikzcd} & (X,S) \arrow[dl] \arrow[dr] & \\ X & & (S,(-K_X)|_S) \end{tikzcd}$$

will give a natural morphism

$$\begin{tikzcd} & \mathcal{P}_g \arrow[dl, "p_g"'] \arrow[dr, "q_g"] & \\ \mathrm{W}_g & & \mathcal{F}_g \end{tikzcd} \tag{3}$$

Question 0.6 *Is there a compactification using K-moduli theory for W_g and $\mathcal{P}_g$ to extend the diagram 3 to be a morphism ? If so, what kind of compactifiaction $\mathcal{F}_g$ should choose ? and what is the boundary morphism ?*

Further problems I plan to explore are

Problem 0.7
1. *Investigate the topological properties of the K-moduli spaces n-dimenional Fano varieties.*
2. *Compute the Picard group of the K-moduli spaces n-dimenional Fano varieties and Investigate its ample cone, effective cone.*
3. *Investigate the relation in $A^*(\mathcal{F}_g)$ from W_g. Can we get new relations in $A^*(\mathcal{F}_g)$ different from those from arithmetic (eg, Borchards relations) and MOP's localization of relative virtual cycles.*

0.3 Cohomology of moduli space of cubic fourfolds

0.3.1 Current work

Let M be the moduli space of cubic fourfolds with ADE singularities at worst. Denote D the period domain and Γ the monodromy group. The locally symmetric space $\Gamma \setminus D$ has Baily-Borel compactification $(\Gamma \setminus D)^*$. The global Torelli theorem identify M as a open subset of $(\Gamma \setminus D)^*$. Based on Kirwan's work, we first compute the cohomology of the partial resolution $\widetilde{\mathcal{M}}$ of the GIT compactifiaction of M,

Theorem 0.8 (Si [10]) *The Poincare polynomial of $\widetilde{\mathcal{M}}$ is given by*

$$\begin{aligned} P_t(\widetilde{\mathcal{M}}) =& 1 + 9t^2 + 26t^4 + 51t^6 + 81t^8 + 115t^{10} + 152t^{12} + 193t^{14} \\ &+ 236t^{16} + 280t^{18} + 324t^{20} + 280t^{22} + 236t^{24} + 193t^{26} \\ &+ 152t^{28} + 115t^{30} + 81t^{32} + 51t^{34} + 26t^{36} + 9t^{38} + t^{40}. \end{aligned}$$

Using decomposition theorem, we also compute the intersection cohomology (middle perversity) of its Baily-Borel compactification

Theorem 0.9 (Si [10]) *The Poincare polynomial of intersection cohomology of* $(\Gamma \setminus D)^*$ *is*

$$\begin{aligned}\mathrm{IP}_t((\Gamma \setminus D)^*) = & 1 + 2t^2 + 4t^4 + 9t^6 + 16t^8 + 26t^{10} + 38t^{12} + 50t^{14} \\ & + 65t^{16} + 82t^{18} + 112t^{20} + 82t^{22} + 65t^{24} + 50t^{26} \\ & + 38t^{28} + 26t^{30} + 16t^{32} + 9t^{34} + 4t^{36} + 2t^{38} + t^{40}.\end{aligned}$$

The cohomology of moduli spaces of cubic threefolds have been computed in [3]. But our case is much complicated.

0.3.2 Future work

To compute the cohomology of $M = \Gamma \setminus D - H_\infty$ and $\Gamma \setminus D$ where H_∞ is a Heegner divisor . I plan to use the result in [11] to determine the kernel of kirwan map. But unluckily, the first two exceptional divisors is not in the setting of [11]. I am trying to overcome this difficulty.

0.4 Cycle theory of Moduli spaces of K3 surfaces

0.4.1 Current work

In [6], Marian-Opera-Pandharipande defined the tautological ring $R^*(\mathcal{F}_g)$ of moduli spaces of K3 surfaces $\mathcal{F}_g$. Let $\mathrm{NL}^*(\mathcal{F}_{2l})$ be the subring generated by Noether-Lefchetz locus. Marian-Opera-Pandharipande make a conjecture that the two subrings are the same. This is proved by Pandharipande- Yin by intersecting WDVV and Getzler relations on $\overline{M}_{0,4}$ and $\overline{M}_{1,4}$ respectively. In their study of $R^*(\mathcal{F}_g)$, Pandharipande- Yin ask the following question for $\mathcal{F}_g$:

Question 0.10 (Pandharipande-Yin) *Do the following results hold ?*

1. $R^{18}(\mathcal{F}_g) = 0,\ R^{19}(\mathcal{F}_g) = 0$;
2. $R^{17}(\mathcal{F}_g) = \mathbb{Q}[\lambda_g^{17}]$.

The question is very hard in general. In joint work in progress with Opera, Pandharipande, Yin, we will give a positive answer to the question 0.10 for $g = 2$ case. Actually, using equivariant chow ring and based on geometry of plane sextic curves, we can determine the ring structure of $A^*(\mathcal{F}_2)$.

0.4.2 Future work

Related conjectures that I am interested are the following

1. **Conjecture 0.11 ([8])** *Let* $\overline{M}_{g,n}(\pi_\Lambda, \beta)$ *be the relative moduli space of stbale maps of the family of K3 surfaces* $(\mathcal{X}, \mathcal{L}_1, \cdots \mathcal{L}_r) \xrightarrow{\pi_\Lambda} \mathcal{F}_\Lambda$ *and* $\overline{M}_{g,n}(\pi_\Lambda, \beta) \xrightarrow{ev} \mathcal{X}^n$ *be the evaluation map, then*

$$ev_*([\overline{M}_{g,n}(\pi_\Lambda, \beta)]^{red}) \in R^*(\mathcal{X}^n).$$

where $R^*(\mathcal{X}^n)$ *is the tautological subring of* $A^*(\mathcal{X}^n)$*.*

The conjecture is open even when $\mathcal{F}_\Lambda$ is a point.

2. **Conjecture 0.12 (Bergeron-Li [2])** *Let* $\mathcal{F}_h$ *be the moduli space of h-polarised IHS* $2n$*-fold, then*

$$R^*(\mathcal{F}_h) = \mathrm{NL}^*(\mathcal{F}_h).$$

3. **Conjecture 0.13 (Bergeron-Li [2])** *Denote* $RH^*(\mathcal{F}_h) := \mathrm{Im}(R^*(\mathcal{F}_h) \xrightarrow{cl} H^*(\mathcal{F}_h))$ *and* $\mathrm{NLH}^*(\mathcal{F}_h) := \mathrm{Im}(\mathrm{NL}^*(\mathcal{F}_h) \xrightarrow{cl} H^*(\mathcal{F}_h))$*. Then*

$$RH^*(\mathcal{F}_h) = \mathrm{NLH}^*(\mathcal{F}_h).$$

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